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d-CRITICAL MODULES OF LENGTH 2 OVER WEYL ALGEBRAS

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ABSTRACT

We construct a family of modules over Weyl algebras with the property of being non-simple of finite length and also d-critical (i.e. $d(M) > d(M/N)$ for every non-trivial submodule N, of M). Here d stands for the Gelfand–Kirillov dimension. We further study some properties of these modules.

Introduction

In the Oberwolfach conference on "Enveloping Algebras" held in February 1985 P. Tauvel asked if any d-critical module of finite length over a solvable Lie-algebra is simple. Here d refers to Gelfand–Kirillov dimension.

Over commutative polynomial rings, the answer to this question is positive. Yet we show that the question has a negative answer already for nilpotent Lie-algebras.

In fact, let $A_n(\mathbb{C})$ be the Weyl algebra of index n over $\mathbb{C}$, that is the algebra generated over $\mathbb{C}$ by $\{p_i, q_i\}_{i=1}^n$ with the relations $[p_i, q_j] = \delta_{ij}$ and $[p_i, p_j] = [q_i, q_j] = 0$ for $i, j = 1, \ldots, n$. Then $A_n(\mathbb{C})$ can be viewed as a quotient of the enveloping algebra of a nilpotent Lie-algebra, e.g. the Heisenberg algebra. It is easy to see that if A_n admits a non-simple d-critical module of finite length then such a module must admit a non-holonomic simple sub-module.

We show that Tauvel's question has a negative answer by adapting Stafford's [St] construction of non-holonomic simple $A_n(\mathbb{C})$ modules.

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Stafford obtained two families of elements of A_n (depending on scalar parameters). The first one in order to obtain simple non-holonomic modules for A_n, and the second one in order to obtain such modules for $U(\mathrm{sl}_2 \times \mathrm{sl}_2)$.

The basic idea here is to "deform" the second of those elements (namely the element $\alpha_\mu := p_2 + q_2p_1q_1 + \lambda q_1(q_1p_1 + \mu) + p_1 \ \in A_2, \ \lambda, \mu \in \mathbb{C}, \lambda \notin \mathbb{Q}, \mu \notin \mathbb{Z}$) to values of the parameter μ for which simplicity fails and then to see what kinds of modules one gets. In order to make this analysis we realized that Stafford's proof of simplicity in the case of the second element was not what we needed. Here we give a different proof based on a careful refinement of Stafford's proof for the simplicity for his first element.

A key point is that for $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ and $\mu \in \{1, 0, -1, -2, \ldots\}$ the element α_μ still generates a maximal ideal of A_2. Here we observe that allowing the μ parameter to take integer ≥ 2 values we get a family of d-critical modules of length 2. In particular this answers Tauvel's question by the negative.

In general it is false that "deformations" of simple non-holonomic modules in the sense of varying the parameters always give modules of finite length. For example if one allows Stafford's λ-parameter (in his second example) to take rational values one gets a module of infinite length.

A further study of the class of modules constructed gave the following interesting results.

First, the simple quotient of each module in this class is holonomic and has multiplicity $(\mu - 1)$ in the sense of the Hilbert-Samuel polynomial. More particularly, for $\mu = 2$, the simple quotient is the standard module for A_2.

Secondly, the simple submodule is, up to an automorphism of A_2, a Stafford module for a "reflected" μ. That is, the new μ parameter takes the value $2 - \mu$. Here we observe a strange analogy with the spectrum of sl_2, although the questions involved are quite different.

In principal one might have answered Tauvel's question by simply showing that there exists a simple holonomic module N and a simple non-holonomic module M with $\mathrm{Ext}^1_{A_n}(M, N) \neq 0$. Unfortunately the main method to calculate such extension groups only works well when the first factor can be presented in the form A_n/I with I a principal ideal, see [Mc-Ro]. This is not possible here. Moreover for non-holonomic modules one has no duality theory which could, for example, make it enough to show that $\mathrm{Ext}^1_{A_n}(N, M) \neq 0$ with N, M as above. In fact, our results show that $\mathrm{Ext}^1_{A_n}(N, M)$ can be non-zero with N a simple holonomic module and M, a simple non-holonomic module over A_n. (For M we take the Stafford module over A_n.)

Although our results hold in the case of A_n for every $n \geq 2$, for greater clarity and brevity we have restricted ourselves to the case of $n = 2$.

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1.

1.1 The standard monomials $p_2^{k_2} p_1^{k_1} q_1^{l_1} q_2^{l_2}$ form a $\mathbb{C}$-basis of A_2, where the quadruple $k := (k_2, k_1, l_1, l_2)$ lies in $\mathbb{N}^4$ and is called the **degree** of $p_2^{k_2} p_1^{k_1} q_1^{l_1} q_2^{l_2}$. Take the lexicographic order on $\mathbb{N}^4$ and define the **degree** of $a \in A_2$ to be the maximal degree of the standard monomials which appear with non-zero coefficient in the expression of a as a sum of such monomials.

For each $a \in A_2$ one defines the map ad a: $b \longmapsto [a, b]$ of A_2 to itself. It is a derivation of A_2.

Set $z := q_1 p_1$. One has $\operatorname{ad} z(p_1^l q_1^k) = (k - l) p_1^l q_1^k$ and we say that $p_1^l q_1^k$ is an **eigenvector** of eigenvalue $(k - l)$ for *ad* z.

1.2 Set $R := \mathbb{C}[q_2, q_1, p_1]$, identified with a subalgebra of A_2.

PROPOSITION: *Let $\lambda, \mu \in \mathbb{C}$ with $\lambda \notin \mathbb{Q}$. Set $\alpha_\mu := p_2 + q_2 p_1 q_1 + \lambda q_1 (q_1 p_1 + \mu) + p_1$. Assume $\beta \in A_2 \backslash \alpha_\mu A_2$ is of minimal degree such that*

$$(1) \qquad I := \alpha_\mu A_2 + \beta A_2 \subsetneq A_2.$$

Then $\beta = c p_1^{\mu - 1}$ for some $c \in \mathbb{C}^*$ and some $\mu \in \{2, 3, \ldots\}$. In particular $\alpha_\mu A_2$ is maximal if $\mu \notin \{2, 3, \ldots\}$.

Proof: The fact that $\alpha_\mu \in R[p_2]$ is monic of degree one in p_2 implies that $\beta \in R$. Write then $\beta = \sum_{i=0}^r p_1^i f_i$ with $f_i \in \mathbb{C}[q_1, q_2]$ and $f_r = \sum_{j=0}^s q_1^j g_j$, $g_j \in \mathbb{C}[q_2]$. We can suppose $f_r \neq 0$, $g_s \neq 0$. If $s > 0$ then by Euclid's algorithm there exist $h_1, h_2 \in \mathbb{C}[q_2]$ such that deg $h_2 <$ deg g_s and $\lambda g_{s-1} = g_s h_1 + h_2$.

Set $v := (r - s) q_2 + \lambda (2r - s) q_1 + h_1$.

The proof is completed in the following four steps.

STEP 1: *Set $\gamma := \alpha_\mu\beta - \beta\alpha_\mu + \beta v \in I$, then $\gamma = 0$.*

This is a consequence of the minimality of β and follows by the direct calculation of [St, Proposition 2.2].

Set $S := \mathbb{C}[q_1, p_1]$, considered as a subalgebra of A_2. One writes β in the form $\beta = \sum_{i=0}^t q_2^i b_i \quad b_i \in S \; b_t \neq 0$.

The equality $\gamma = 0$ then gives:

$$\sum_{i=0}^{t}\{iq_2^{i-1}b_i + q_2^{i+1}[p_1q_1, b_i] + q_2^i[\lambda q_1(q_1p_1 + \mu) + p_1, b_i] \tag{2}$$
$$+ (r-s)q_2^{i+1}b_i + \lambda(2r-s)q_2^i b_i q_1\} = -h_1(\sum_{i=0}^t q_2^i b_i).$$

All the terms in the L.H.S. have degree $\leq t+1$ in q_2. Thus deg $h_1 \leq 1$. One writes then $h_1 = k_1q_2 + k_2, \quad k_1, k_2 \in \mathbb{C}$.

STEP 2: *Set $u := r - s + k_1$, then b_t is an eigenvector of ad z with eigenvalue $-u$. In particular $u \in \mathbb{Z}$ and one can write b_t in the form $b_t = \sum a_i p_1^{i+u} q_1^i \;\; a_i \in \mathbb{C}$.*

Inspection of the coefficient of q_2^{t+1} in (2) gives: $[p_1q_1, b_t] + ub_t = 0$, $b_t \in S$ and the result follows.

STEP 3: *Either $t > 0$ or $\beta = cp_1^{\mu-1}$ for some $c \in \mathbb{C}^*$, $\mu \in \{2, 3, \ldots\}$.*

Suppose that $t = 0$. Then in (2) every term is an eigenvector of *ad* z, and so the sum of all the eigenvectors with the same eigenvalue is 0. In particular $k_2b_0 = 0$ from the $-u$ eigenvalue and then $k_2 = 0$ because $b_0 \neq 0$ by hypothesis. Again $[p_1, b_0] = 0$ from the $-u-1$ eigenvalue which implies $b_0 \in \mathbb{C}[p_1]$.

Yet b_0 is an eigenvector of *ad* z with eigenvalue $-u$ so $b_0 = cp_1^u$ for some $c \in \mathbb{C}\backslash\{0\}$.

Finally $\lambda([q_1^2p_1 + \mu q_1, p_1^u] + (2r-s)p_1^u q_1) = 0$ from the $-u+1$ eigenvalue. Since $\lambda \notin \mathbb{Q}$ by hypothesis we have in particular that $\lambda \neq 0$. A calculation then gives that

$$-2up_1^u q_1 - u(-u+\mu-1)p_1^{u-1} + (2r-s)p_1^u q_1 = 0.$$

This gives $\mu = u + 1$. Yet $\beta = b_0 = cp_1^u$ so that u is a strictly positive integer. This gives $\mu \in \{2, 3, \ldots\}$.

STEP 4: *If $t > 0$, then $I = A_2$.*

We shall use the condition $\lambda \notin \mathbb{Q}$. The coefficient of q_2^t in the equation (2) is:

$$[p_1q_1, b_{t-1}] + ub_{t-1} + \lambda([q_1^2p_1 + \mu q_1, b_t] + (2r-s)b_tq_1) + [p_1, b_t] + k_2b_t = 0. \tag{3}$$

We develop b_{t-1} as a sum of eigenvectors of $ad\ z$ and we collect together all the terms with the same eigenvalue.

We remark that $[q_1p_1, b_{t-1}] + ub_{t-1}$ cannot have a term of eigenvalue $-u$, so by (3) we obtain $k_2 = 0$.

From (3) we see that $b_{t-1} = f_1 + f_2 + f_3$ where f_1, f_2, f_3, which correspond respectively to the eigenvalues: $-u, -u+1, -u-1$ of $ad\ z$. Furthermore f_1, f_2, f_3 must satisfy the following equations:

$[p_1q_1, f_1] + uf_1 = 0,$

$[p_1q_1, f_2] + uf_2 + \lambda([q_1^2p_1 + \mu q_1, b_t] + (2r-s)b_tq_1) = 0,$

$[p_1q_1, f_3] + uf_3 + [p_1, b_t] = 0.$

Equating the coefficient of q_2^{t-1} in (2) to zero gives:

$$(4)\qquad \begin{aligned} &tb_t + [p_1q_1, b_{t-2}] + ub_{t-2} \\ &\quad + \lambda\{[q_1^2p_1 + \mu q_1, b_{t-1}] + (2r-s)b_{t-1}q_1 + [p_1, b_{t-1}]\} = 0 \end{aligned}$$

View b_t as a sum of monomials in S. Let b_t^* denote the monomial in b_t of the highest degree (viewed as an element of A_2).

Collecting terms in (4) corresponding to the eigenvalue $-u$ and of highest degree and using the above expression for b_{t-1} we obtain:

$$tb_t^* - \lambda\{[q_1^2p_1, [p_1, b_t^*]] + (2r-s)([p_1, b_t^*]q_1 + [p_1, b_t^*q_1]) + [p_1, [q_1^2p_1, b_t^*]]\} = 0.$$

The term in brackets is an integral multiple of b_t^* and as $t \neq 0$ (by Step 3) this gives $\lambda \in \mathbb{Q}$ in contradiction to the hypothesis on λ.

We conclude that $I = A_2$ as required.

1.3 From now on take $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ and $\mu \in \{2, 3, \ldots\}$. Via the lexicographic ordering we have $\alpha_\mu A_2 \bigcap R = \{0\}$.

Lemma 1: *Set* $\alpha_\mu := p_2 + q_2p_1q_1 + \lambda(q_1^2p_1 + \mu q_1) + p_1$. *Then* $\alpha_\mu A_2$ *is not a maximal ideal of* A_2.

Proof: An elementary calculation gives:

$$(5)\qquad [\alpha_\mu, p_1^{\mu-1}] + p_1^{\mu-1}v = 0$$

where $v := (\mu-1)q_2 + 2\lambda(\mu-1)q_1$. Suppose that $\alpha_\mu A_2$ is a maximal ideal. Obviously $p_1^{\mu-1} \notin \alpha_\mu A_2$ for $\mu \geq 1$. Hence there exist $\epsilon_1, \epsilon_2 \in A_2$ such that $\alpha_\mu\epsilon_1 + p_1^{\mu-1}\epsilon_2 = 1$, that is $1 - p_1^{\mu-1}\epsilon_2 = \alpha_\mu\epsilon_1$. Suppose that $\epsilon_2 \in R$. Then from

$\alpha_\mu A_2 \bigcap R = \{0\}$ we obtain $1 - p_1^{\mu-1}\epsilon_2 = 0$, since $\mu \geq 2$ this would imply that p_1 is invertible in A_2, which is false. We can write $\epsilon_2 = p_2\beta_1 + \beta_2$ with $\beta_2 \in R$, $\beta_1 \in A_2$ of strictly lower degree in p_2 than ϵ_2. Then by (5) we obtain

$$1 - p_1^{\mu-1}(\epsilon_2 - (\alpha_\mu - v)\beta_1) = 1 - p_1^{\mu-1}\epsilon_2 + \alpha_\mu p_1^{\mu-1}\beta_1 = \alpha_\mu(\epsilon_1 + p_1^{\mu-1}\beta_1)$$

which gives $\alpha_\mu\epsilon_1' + p_1^{\mu-1}\epsilon_2' = 1$ where $\epsilon_1' = \epsilon_1 + p_1^{\mu-1}\beta_1$, $\epsilon_2' = \epsilon_2 - (\alpha_\mu - v)\beta_1$. Yet ϵ_2' has a strictly lower degree in p_2 than ϵ_2, so we can assume $\epsilon_2 \in R$, without loss of generality. However by our first observation this leads to a contradiction.

Combining Lemma 1 with Proposition 1.2 it follows that $I := \alpha_\mu A_2 + p_1^{\mu-1}A_2$ is the unique right ideal satisfying $\alpha_\mu A_2 \subsetneq I \subsetneq A_2$. This can be expressed as follows:

LEMMA 2: *The right $A_2/\alpha_\mu A_2$ is uniserial of length 2.*

LEMMA 3: *$A_2/\alpha_\mu A_2$ is d-critical.*

Proof: By [Kr-Le, cor 9.6] $d(A_2/\alpha A_2) = 3$.

On the other hand $A_2/(\alpha_\mu A_2 + p_1^{\mu-1}A_2)$ is a finitely generated $\mathbb{C}[q_1, q_2]$-module and $d_{A_2}(A_2/(\alpha_\mu A_2 + p_1^{\mu-1}A_2)) = d_{\mathbb{C}[q_1,q_2]}(A_2/(\alpha_\mu A_2 + p_1^{\mu-1}A_2))$. Also, since $A_2/(\alpha_\mu A_2 + p_1^{\mu-1}A_2)$ is a finitely generated $\mathbb{C}[q_1, q_2]$-module, the GK dimension of the latter is ≤ 2, for example by [Kr-Le, Lemma 8.1]. This shows that $d_{A_2}(A_2/(\alpha_\mu A_2 + p_1^{\mu-1}A_2)) = 2$ as required.

1.4 Let θ_μ denote the automorphism of A_2 satisfying: $\theta_\mu(p_2) = p_2 - (\mu - 1)q_2$ and fixing R. By Proposition 1.2 we get that $\alpha_{2-\mu}A_2$ is a right maximal ideal of A_2. Retain the notations of 1.3 and recall that $v := (\mu - 1)q_2 + 2\lambda(\mu - 1)q_1$. One has $\theta_\mu(\alpha_{2-\mu}) = \alpha_\mu - v$, so that $(\alpha_\mu - v)A_2$ is a maximal right ideal of A_2.

LEMMA: $\operatorname{soc}(A_2/\alpha_\mu A_2) \cong A_2/\alpha_{2-\mu}A_2$

Proof: By 1.3 we get that

$$\operatorname{soc}(A_2/\alpha_\mu A_2) = (\alpha_\mu A_2 + p_1^{\mu-1}A_2)/\alpha_\mu A_2 \cong p_1^{\mu-1}A_2/p_1^{\mu-1}\bigcap \alpha_\mu A_2 =: N.$$

Consider $p_1^{\mu-1}$ as a generator of N. By (5) we have $[\alpha_\mu, p_1^{\mu-1}] = -p_1^{\mu-1}v$. Thus $\alpha_\mu - v \in \operatorname{Ann}_{A_2}(p_1^{\mu-1})$, so that by our previous observation $\theta_\mu(\alpha_{2-\mu})A_2 \subset \operatorname{Ann}_{A_2}(p_1^{\mu-1})$. Yet $\theta_\mu(\alpha_{2-\mu})A_2$ is a maximal right ideal of A_2, while $\operatorname{Ann}_{A_2}(p_1^{\mu-1})$ is a proper right ideal. Hence equality holds, and this proves our assertion.

Remark: Here we see the the analogy with the spectrum of $U(\mathrm{sl}_2)$ alluded to in the introduction.

1.5 Let $e(M)$ denote the multiplicity of the module M.

LEMMA: *We have* $e(A_2/(\alpha_\mu A_2 + p_1^{\mu-1}A_2)) = \mu - 1$.

Proof: Let $\{U_n\}_{n\in\mathbb{N}}$ denote the canonical filtration on A_2 i.e. $U_0 = \mathbb{C}, U_1 = \mathbb{C}p_1 + \mathbb{C}p_2 + \mathbb{C}q_1 + \mathbb{C}q_2$ and $U_n = (U_0 + U_1)^n$ for $n \geq 2$. Clearly $M_0 := A_2/(\alpha_\mu A_2 + p_1^{\mu-1}A_2)$ is generated by $W := \{\bar{1}, \bar{p_1}, \ldots, \bar{p_1}^{\mu-2}\}$ where $\bar{x}$ denotes the image of $x \in A_2$ under the canonical projection of A_2 onto M. Let us compute bounds on $\dim_{\mathbb{C}} WU_n$.

Suppose that the map $w \otimes x \longmapsto wx$ from $W \bigotimes \mathbb{C}[q_1, q_2]$ to M_0 is not injective. Then we have $f = \sum_{i=0}^{\mu-2} p_1^i g_i$, $g_i \in \mathbb{C}[q_1, q_2]$ whose image is zero in M_0, that is $f \in \alpha_\mu A_2 + p_1^{\mu-1}A_2$. Yet $\deg f < \deg p_1^{\mu-1}$, so this contradicts Proposition 1.2. We conclude that

$$\dim_{\mathbb{C}} WU_n \geq \frac{(\mu-1)}{2} n(n+1). \tag{6}$$

Let V_n denote the space of polynomials of degree $\leq n$ in $\mathbb{C}[q_1, q_2]$. We show that $p_2^n \in WV_{n+\mu-2} \mod (\alpha_\mu A_2 + p_1^{\mu-1}A_2)$ for all $n \in \mathbb{N}$. This implies that

$$\dim_{\mathbb{C}} WU_n \leq \frac{(\mu-1)}{2}(n+\mu-1)(n+\mu-2).$$

Combined with the opposite inequality (6) this will establish the assertion of the lemma.

Calculating $\mod (\alpha_\mu A_2 + p_1^{\mu-1}A_2)$ we have $p_2 = -q_2p_1q_1 - \lambda q_1^2 p_1 - \lambda\mu q_1 - p_1 = -p_1(q_2q_1 + \lambda q_1^2 + 1) + (2 - \lambda\mu)q_1 \in p_1V_2 + V_1$.

From the inclusion $V_m p_1 \subset p_1 V_m + V_{m-1}$ we obtain $\mod (\alpha_\mu A_2 + p_1^{\mu-1}A_2)$ that $p_2^n \in (p_1V_2 + V_1)^n \subset \sum_{i=0}^{\mu-2} p_1^i V_{n+i} \subset WV_{n+\mu-2}$ as required.

Remark: By 1.3 we see that $(\alpha_\mu A_2 + p_1^{\mu-1}A_2)/\alpha_\mu A_2$ is the only non-trivial submodule of $A_2/\alpha_\mu A_2$. Hence $A_2/\alpha_\mu A_2$ is a non-trivial extension of the simple non-holonomic module $M := A_2/(\alpha_\mu A_2 + p_1^{\mu-1}A_2)$.

Consequently $\mathrm{Ext}^1_{A_2}(M, N) \neq 0$. For $\mu = 2$ we get that $M \cong \mathbb{C}[q_1, q_2]$ as an A_2-module, that is M is isomorphic to the so called standard module over A_2.

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